TN 2 & 3: Higher order approx.

Recall: *1st Taylor polynomial*

$$T_1(x) = f(b) + f'(b)(x-b)$$

Error Bound

On interval [a,b], if $|f''(x)| \leq M$,

then $|f(x) - T_1(x)| \leq \frac{M}{2}|x-b|^2$.

Entry Task: Let $f(x) = x^{1/3}$.

(a) Find the 1st Taylor Polynomial based at $b = 8$.

(b) Give a bound on the error over the interval $[7,9]$.

$f(x) = x^{1/3} \rightarrow f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \rightarrow f'(8) = \frac{1}{3\cdot 4} = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}$

$$\boxed{T_1(x) = 2 + \frac{1}{12}(x-8)}$$

THUS, $x^{1/3} \approx 2 + \frac{1}{12}(x-8)$ for x CLOSE TO 8.

ERROR BOUND

DECREASING ON [7,9]

$|f''(x)| = \frac{2}{9x^{5/3}} \leq \frac{2}{9(7)^{5/3}} \approx 0.008675 = M$

9 (or 7)

$\Rightarrow$ ERROR $\leq \frac{0.008675}{2}|x-8|^2$

$\approx \boxed{0.0043377}$ ← ERROR BOUND

$\boxed{\text{EXAMPLE}}$ $\sqrt[3]{9} \approx 2 + \frac{1}{12}(9-8)$

$= 2 + \frac{1}{12} = \boxed{2.08\overline{3}} \pm 0.004$

"ACTUAL" = 2.0800838...

2^{nd} Taylor Polynomial is given by

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2} f''(b)(x-b)^2$$

Quadratic error bound theorem

On interval [a,b], if $|f'''(x)| \le M$, then $|f(x) - T_2(x)| \le \frac{M}{6}|x-b|^3$.

Example:

Find the 2^{nd} Taylor polynomial for $f(x) = x^{1/3}$ based at b = 8 and find an error bound on the interval [7,9].

$$f''(8) = -\frac{2}{9(8)^{5/3}} = -\frac{2}{9 \cdot 32} = -\frac{1}{144}$$

$$T_2(x) = 2 + \frac{1}{12}(x-8) + \frac{1}{2}\left(-\frac{1}{144}\right)(x-8)^2$$

$$= \underbrace{2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2}$$

$$x^{1/3} \simeq 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

for x CLOSE TO 8

ERROR BOUND

$$f'''(x) = \frac{10}{27} x^{-8/3} = \frac{10}{27x^{8/3}}$$

$$|f'''(x)| \le \frac{10}{27(7)^{8/3}} \approx 0.002065577$$

$$\text{Error} \le \frac{0.0020656}{6} |x-8|^3$$

9 (or 7)

$$= 0.000344$$

EXAMPLES

$$\sqrt[3]{9} \approx 2 + \frac{1}{12}(9-8) - \frac{1}{288}(9-8)^2$$

$$\approx \boxed{2.079861} \pm 0.000344$$

"ACTUAL" = 2.0800838

Taylor Approximation Idea:

If two functions have ***all*** the same derivative values, then they are the same function (up to a constant).

Let's compare derivatives of $f(x)$ and $T_2(x)$ at b.

$$
\begin{aligned}
T_2(x) &= f(b) + f'(b)(x-b) + \tfrac{1}{2} f''(b)(x-b)^2 \quad + \quad f'''(b)(x-b)^3 \\
T_2'(x) &= 0 \quad + f'(b) \quad + f''(b)(x-b) \leftarrow \text{CANCEL} \qquad \tfrac{1}{3} f'''(b)\,3(x-b)^2 \\
T_2''(x) &= 0 \quad + 0 \quad + f''(b) \qquad \tfrac{1}{2}\tfrac{1}{3} f'''(b)\,3\cdot 2\,(x-b) \\
T_2'''(x) &= 0
\end{aligned}
$$

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Now plug in x = b to each of these.

- What do you see?
- Why did we need a ½ ?
- What would $T_3(x)$ look like?
- What would $T_4(x)$ look like? ($T_5(x)$?, $T_6(x)$?...)

$$+ \frac{1}{3!} f'''(b)(x-b)^3$$

$$+ \frac{1}{4!} f^{(4)}(b)(x-b)^4$$

$$+ \frac{1}{5!} f^{(5)}(b)(x-b)^5$$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

$0! = 1$ ← DEFIN

$1! = 1$

$2! = 2$

n^{th} Taylor polynomial

$$f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$$

In sigma notation:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

Taylor's Inequality (error bound):
on a given interval [a,b],
if $|f^{(n+1)}(x)| \le M$, then

$$|f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-b|^{n+1}$$

Example: Find the 9^{th} Taylor polynomial for $f(x) = e^x$ based at $b = 0$, and give an error bound on the interval $[-2,2]$.

$f(x) = e^x \to f(0) = 1$

$f'(x) = e^x \to f'(0) = 1$

$f''(x) = e^x \to f''(0) = 1$

$f^{(9)}(x) = e^x \to f^{(9)}(0) = 1$

$1 + 1(x-0) + \frac{1}{2!}1(x-0)^2 + \frac{1}{3!}1(x-0)^3 + \cdots + \frac{1}{9!}1\cdot(x-0)^9$

$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{9!}x^9$

ERROR BOUND

INCREASING FUNCTION

MAX AT $x = 2$

$f^{(10)}(x) = e^x \le e^2 = M$

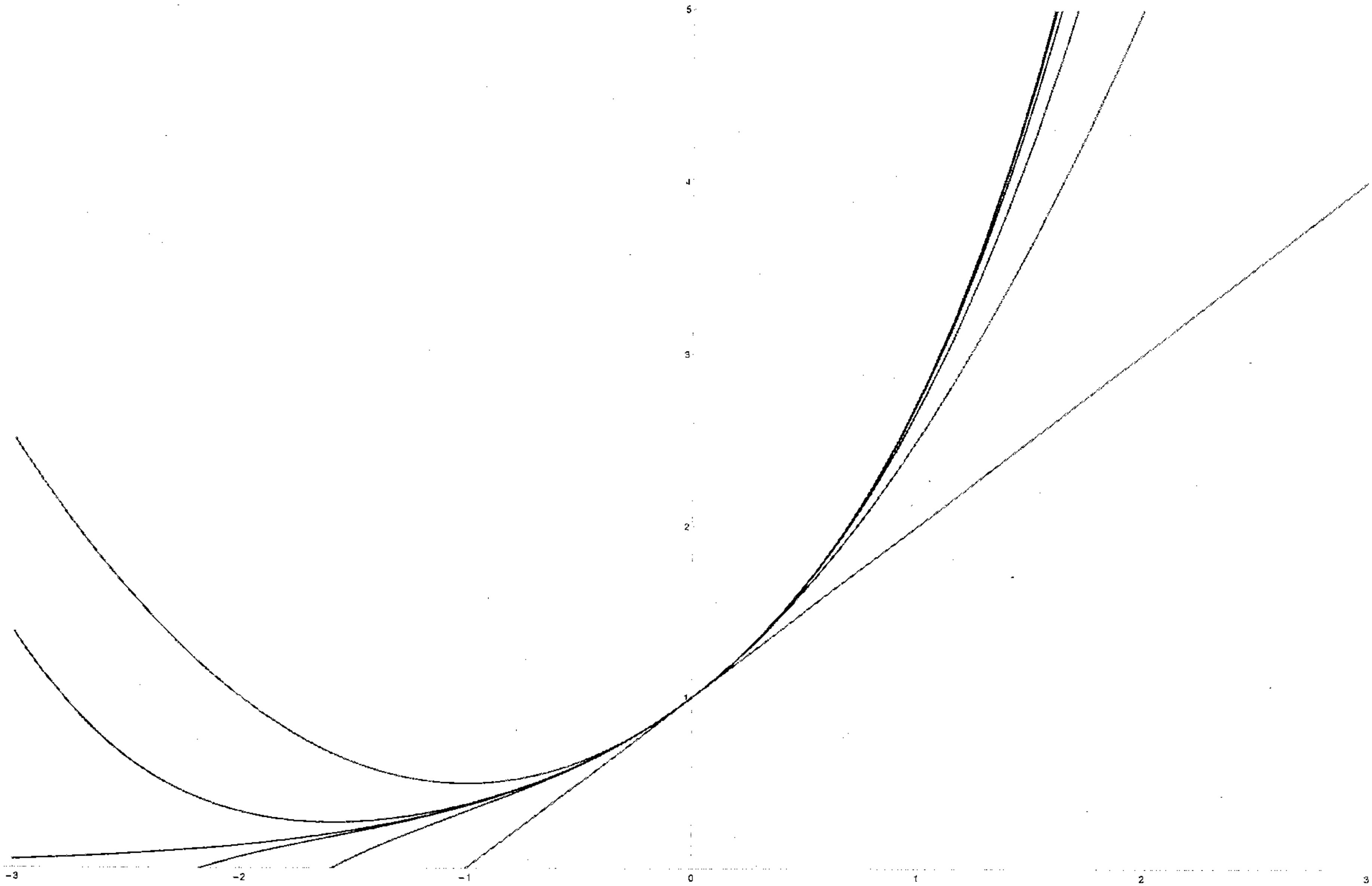
ERROR $\le \frac{e^2}{10!}|x-0|^{10} \le \frac{e^2}{10!}2^{10} \approx 0.002085$

EXAMPLE.

$e^{1.8} \approx 1 + (1.8) + \frac{1}{2!}(1.8)^2 + \cdots + \frac{1}{9!}(1.8)^9$ ± 0.002

$f(x) = e^x$ and

$T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, $T_5(x)$



Example: Again consider,

$$f(x) = e^x \text{ based at } b = 0$$

Find the first value of *n* when Taylor's inequality gives an error less than 0.0001 on $[-2,2]$.

WANT $\frac{e^2}{(n+1)!} |\overset{2}{x} - 0|^{n+1} \leq 0.0001$

THERE IS NO GOOD WAY TO SOLVE THIS EXACTLY.

SO WE GUESS AND CHECK!

$n = 9 \Rightarrow$ NO (WE JUST DID IT AND ERROR WAS 0.004)

$n = 10 \Rightarrow$ ERROR $\leq \frac{e^2}{11!} 2^{11} \approx 0.000379$ NOT SMALL ENOUGH YET

$n = 11 \Rightarrow$ ERROR $\leq \frac{e^2}{12!} 2^{12} \approx 0.000063$ YES!

$\boxed{n = 11}$

Side Note:

For a fixed constant, *a*, the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity.

So the expression $\frac{1}{(n+1)!}|x-b|^{n+1}$, will always go to zero as *n* gets bigger.

Which means that the error goes to zero, unless something unusual is happening with M, which will see in examples later.

TN 4: Taylor Series

Def'n: The **Taylor Series** for f(x) based at b is defined by

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$$

If the limit exists at *x*,
then we say it **converges** at x.
(*i.e.* the error goes to zero at x)

Otherwise, we say it **diverges** at x.

The **open interval of convergence** gives the largest open interval over which the series converges.

Note: If

$$\lim_{n\to\infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then *x* is in the open interval of convergence.